Linear systems, row reduction and echelon forms: Part 1

We begin with the essential vocabulary for this topic.

For sake of example, here is a linear system of 3 equations in 3 unknowns.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

Definition: (Coefficient matrix and augmented matrix) The matrix whose columns are the coefficients of each variable in a linear system is called the *coefficient matrix.* The coefficient matrix with one more column containing the values on the right hand side of a linear system is called the *augmented matrix* of the system.

For example, the coefficient matrix of the linear system above is below to the left and the augmented matrix is below to the right.

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

Definition: (Elementary row operations)

1. (*Replacement*) Replace one row by the sum of itself and a multiple of another row.
2. (*Interchange*) Interchange two rows.
3. (*Scaling*) Multiply all entries in a row by a nonzero constant.

Note that these align with the valid operations on systems of equations while preserving the solution set. Because this is the case, the following definition is worth considering.

Definition: (Row equivalent) Two matrices are *row equivalent* if there is a sequence of elementary row operations that transforms one matrix into the other. We use $\sim$ to denote row equivalence between two matrices.

For example,

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 \\ 0 & 2 & -8 \\ 1 & 0 & -1 \end{bmatrix}.$$

That is, these two matrices are row equivalent as the left can be transformed into the right by an application of replacement ($R1 \leftrightarrow R1 + R2$) followed by an application of scaling ($\frac{1}{5}R3$).

Definition: (Leading entry) The *leading entry* of a (nonzero) row of a matrix is the leftmost nonzero entry in that row.

Definition: (Echelon forms) A matrix is in *echelon form* if satisfies the following

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

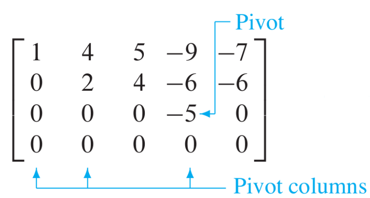
A matrix in echelon form is in *reduced echelon form* (or *reduced row echelon form*) if it additionally satisfies the following

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

For example, the matrices below are in echelon form. The right matrix is in reduced echelon form.

$$\begin{bmatrix} 2 & -3 & 2 & 7 \\ 0 & 5 & 1 & 46 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 13 \\ 0 & 1 & 0 & 17 \\ 0 & 0 & 1 & 19 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition: (Pivots) A *pivot position* (or simply *pivot*) is a location in a matrix that corresponds to a leading 1 in the reduced echelon form of that matrix. A *pivot column* is a column of a matrix that contains a pivot position.



Definition: (Basic and free variables) The variables corresponding to pivot columns of a matrix are called *basic variables*. All other variables are *free variables*.

For example, the previous matrix corresponds to the linear system

$$\begin{aligned} x_1 + 4x_2 + 5x_3 - 9x_4 &= -7 \\ 2x_2 + 4x_3 - 6x_4 &= -6 \\ -5x_4 &= 0 \\ 0 &= 0 \end{aligned}$$

Thus, the basic variables are x_1, x_2 and x_4. Therefore x_3 is a free variable.

Definition: (Parametric descriptions) A *parametric description* of a solution set has all basic variables given in terms of free variables.

The idea here is that the free variables are parameters which we are "free" to choose the value of. For example, from above we have

$$\begin{cases} x_1 &= 17 + 11x_3 \\ x_2 &= -6 - 4x_3 \\ x_4 &= 0 \end{cases}$$

is the parametric description of the solution set to that linear system. Any choice of x_3 yields a solution to the system. For instance, if we choose $x_3 = 0$, then $(17, -6, 0, 0)$ is a particular solution of the system.